DAY TWENTY FOUR

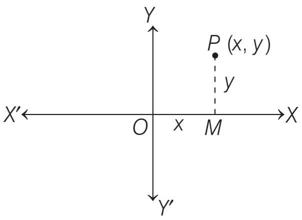
Cartesian System of Rectangular Coordinates

Learning & Revision for the Day

- Rectangular Coordinates
- Distance Formula
- Section Formulae
- Area of a Triangle
- Area of Some Geometric Figures
- Coordinates of Different Points of a Triangle
- Translation of Axes
- Slope of a Line
- Locus and its Equation

Rectangular Coordinates

Let XOX' and YOY' be two perpendicular axes in the plane intersecting at O (as shown in the figure). Let P be any point in the plane. Draw PM perpendicular to OX.



The ordered pair (x, y) is called the rectangular or cartesian **coordinates of point** P.

Distance Formula

Let $P(x_1, y_1)$ and $Q(x_2, y_2)$ be the two points.

Then, $PQ = d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$ or $\sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$

Distance between the points $(0, 0)$ and (x, y) is $\sqrt{x^2 + y^2}$.

Section Formulae

The coordinates of a point which divide the line segment joining two points $P(x_1, y_1)$ and $Q(x_2, y_2)$ in the ratio $m_1 : m_2$ are

(i) $\left(\dfrac{m_1 x_2 + m_2 x_1}{m_1 + m_2}, \dfrac{m_1 y_2 + m_2 y_1}{m_1 + m_2}\right)$ [internal division]

(ii) $\left(\dfrac{m_1 x_2 - m_2 x_1}{m_1 - m_2}, \dfrac{m_1 y_2 - m_2 y_1}{m_1 - m_2}\right)$ [external division]

When m_1 and m_2 are of opposite signs, then division is **external**.

- Mid-point of the line joining (x_1, y_1) and (x_2, y_2) is $\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$.
- Coordinates of any point on one line segment which divide the line segment joining two points $P(x_1, y_1)$ and $Q(x_2, y_2)$ in the ratio $\lambda : 1$ are given by

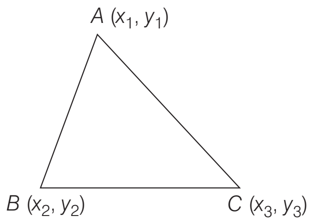
$$\left(\frac{x_1 + \lambda x_2}{\lambda + 1}, \frac{y_1 + \lambda y_2}{\lambda + 1}\right), (\lambda \neq -1)$$

- X-axis and Y-axis divide the line segment joining the points (x_1, y_1) and (x_2, y_2) in the ratio of $-\frac{y_1}{y_2}$ and $-\frac{x_1}{x_2}$ respectively.

If the ratio is positive, then the axis divides it internally and if ratio is negative, then the axis divides externally.

Area of a Triangle

Area of a triangle whose three vertices has coordinates $(x_1, y_1), (x_2, y_2)$ and (x_3, y_3) as shown in the figure below is given by



$$\text{Area of a } \Delta ABC = \frac{1}{2}\left|\begin{matrix}(x_1 y_2 + x_2 y_3 + x_3 y_1) \\ -(y_1 x_2 + y_2 x_3 + y_3 x_1)\end{matrix}\right|$$

$$= \frac{1}{2}|x_1(y_2 - y_3) + x_2(y_3 - y_1) + x_3(y_1 - y_2)| = \frac{1}{2}\left|\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}\right|$$

It should be noted that area is a positive quantity and its unit is square of unit of length.

In the inverse problems, i.e. when area of a triangle is given to be a square units, then we have

$$\Delta ABC = \frac{1}{2}\left|\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}\right| = a \Rightarrow \frac{1}{2}\left|\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}\right| = \pm a$$

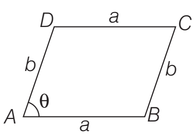
NOTE If area of ΔABC is zero. It mean points are collinear.

Area of Some Geometrical Figures

(i) Suppose a and b are the adjacent sides of a parallelogram and θ be the angle between them as shown in the figure below, then area of parallelogram $ABCD = ab \sin\theta$.

(ii) Area of convex quadrilateral with vertices $(x_1, y_1), (x_2, y_2), (x_3, y_3), (x_4, y_4)$ in that order is

$$\frac{1}{2}\begin{vmatrix} x_1 - x_3 & y_1 - y_3 \\ x_2 - x_4 & y_2 - y_4 \end{vmatrix}.$$



(iii) A triangle having vertices $(at_1^2, 2at_1), (at_2^2, 2at_2)$ and $(at_3^2, 2at_3)$, then area of triangle $= a^2[(t_1 - t_2)(t_2 - t_3)(t_3 - t_1)]$

(iv) Area of triangle formed by coordinate axes and the lines $ax + by + c$ is $= \frac{c^2}{2ab}$.

Coordinates of Different Points of a Triangle

1. Centroid

The centroid of a triangle is the point of intersection of its medians. It divides the medians in the ratio 2 : 1. If $A(x_1, y_1)$, $B(x_2, y_2)$ and $C(x_3, y_3)$ are the vertices of ΔABC, then the coordinates of its centroid G are $\left(\frac{x_1 + x_2 + x_3}{3}, \frac{y_1 + y_2 + y_3}{3}\right)$

2. Orthocentre

The orthocentre of a triangle is the point of intersection of its altitudes. If $A(x_1, y_1)$, $B(x_2, y_2)$ and $C(x_3, y_3)$ are the vertices of a ΔABC, then the coordinates of its orthocentre O are

$$\left(\frac{x_1 \tan A + x_2 \tan B + x_3 \tan C}{\tan A + \tan B + \tan C}, \frac{y_1 \tan A + y_2 \tan B + y_3 \tan C}{\tan A + \tan B + \tan C}\right)$$

3. Circumcentre

The circumcentre of a triangle is the point of intersection of the perpendicular bisectors of its sides. It is the centre of the circle passing through the vertices of a triangle and so it is equidistant from the vertices of the triangle.

Here, $OA = OB = OC$, where O is the centre of circle and A, B and C are the vertices of a triangle. The coordinates of the circumcentre are also given by

$$S\left(\frac{x_1 \sin 2A + x_2 \sin 2B + x_3 \sin 2C}{\sin 2A + \sin 2B + \sin 2C}, \frac{y_1 \sin 2A + y_2 \sin 2B + y_3 \sin 2C}{\sin 2A + \sin 2B + \sin 2C}\right)$$

Incentre

The point of intersection of the internal bisectors of the angles of a triangle is called its incentre.

If $A(x_1, y_1)$, $B(x_2, y_2)$ and $C(x_3, y_3)$ are the vertices of a ΔABC such that $BC = a, CA = b$ and $AB = c$, then the coordinates of the incentre are $I\left(\frac{ax_1 + bx_2 + cx_3}{a + b + c}, \frac{ay_1 + by_2 + cy_3}{a + b + c}\right)$.

Excentre

Coordinate of excentre opposite of $\angle A$ is given by

$I_1 \equiv \left(\frac{-ax_1 + bx_2 + cx_3}{-a + b + c}, \frac{-ay_1 + by_2 + cy_3}{-a + b + c}\right)$ and similarly for excentres (I_2 and I_3) opposite to $\angle B$ and $\angle C$ are given by

$$I_2 \equiv \left(\frac{ax_1 - bx_2 + cx_3}{a - b + c}, \frac{ay_1 - by_2 + cy_3}{a - b + c}\right)$$

and $I_3 \equiv \left(\frac{ax_1+bx_2-cx_3}{a+b-c}, \frac{ay_1+by_2-cy_3}{a+b-c}\right)$.

In an equilateral triangle, orthocentre, centroid, circumcentre, incentre, coincide.

Important Results

- Circumcentre of the right angled ΔABC, right angled at A is $\frac{B+C}{2}$.
- Orthocentre of the right angled ΔABC, right angled at A is A.
- Orthocentre, centroid, circumcentre of a triangle are collinear.
- Centroid divides the line joining the orthocentre and circumcentre in the ratio 2 : 1.
- The circumcentre of right angled triangle is the mid-point of the hypotenuse.
- A triangle is isosceles, if any two of its medians are equal.

Translation of Axes

1. To Alter the Origin of Coordinates Without Altering the Direction of the Axes

Let origin $O(0,0)$ be shifted to a point (a, b) by moving the X and Y-axes parallel to themselves. If the coordinates of point P with reference to old axis are (x_1, y_1), then coordinates of this point with respect to new axis will be $(x_1 - a, y_1 - b)$.

2. To Change the Direction of the Axes of Coordinates without Changing Origin

Let OX and OY be the old axes and OX' and OY' be the new axes obtained by rotating the old OX and OY through an angle θ, then the coordinates of $P(x, y)$ with respect to new coordinate axes will be given by

	$x \downarrow$	$y \downarrow$
$x' \rightarrow$	$\cos\theta$	$\sin\theta$
$y' \rightarrow$	$-\sin\theta$	$\cos\theta$

(i) x and y are old coordinates, x', y' are new coordinates.

(ii) The axes rotation in anti-clockwise is positive and clockwise rotation of axes is negative.

3. To Change the Direction of the Axes of Coordinates by Changing the Origin

If $P(x, y)$ and the axes are shifted parallel to the original axis, so that new origin is (α, β) and then the axes are rotated about the new origin (α, β) by angle ϕ in the anti-clockwise (x', y'), then the coordinates of P will be given by

$$x = \alpha + x'\cos\phi - y'\sin\phi$$
$$y = \beta + x'\sin\phi + y'\cos\phi$$

Slope of a Line

The tangent of the angle that a line makes with the positive direction of the X-axis is called the **slope** or **gradient** of the line. The slope of a line is generally denoted by m.

Thus, $m = \tan\theta$.

Slope of a Line in Terms of Coordinates of any Two Points on it

Let $P(x_1, y_1)$ and $Q(x_2, y_2)$ be two points on a line making an angle θ with the positive direction of X-axis. Then, its slope m is given by

$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{\text{Difference of ordinates}}{\text{Difference of abscissa}}$$

Parallel and Perpendicular Lines on the Coordinate Axes

A line parallel to X-axis makes an angle of $0°$ with X-axis. Therefore, its slope is $\tan 0° = 0$. A line parallel to Y-axis i.e. perpendicular to X-axis makes an angle of $90°$ with X-axis, so its slope is $\tan\frac{\pi}{2} = \infty$. Also, the slope of a line equally inclined with axes is 1 or -1 as it makes an angle of $45°$ or $135°$ with X-axis.

Locus and its Equation

It is the path or curve traced by a moving point satisfying the given condition.

Equation to the Locus of a Point

The equation to the locus of a point is the algebraic relation which is satisfied by the coordinates of every point on the locus of the point.

Steps to Find the Locus of a Point

The following steps are used to find the locus of a point

Step I Assume the coordinates of the point say (h, k) whose locus is to be find.

Step II Write the given condition involving (h, k).

Step III Eliminate the variable(s), if any.

Step IV Replace $h \rightarrow x$ and $k \rightarrow y$. The equation, so obtained is the locus of the point which moves under some definite conditions.

DAY PRACTICE SESSION 1

FOUNDATION QUESTIONS EXERCISE

1 Length of the median from B to AC where $A(-1, 3)$, $B(1, -1)$, $c(5, 1)$ is

(a) $\sqrt{18}$ (b) $\sqrt{10}$ (c) $2\sqrt{3}$ (d) 4

2 Three points $(p+1, 1)$, $(2p+1, 3)$ and $(2p+2, 2p)$ are collinear if p is equal to

(a) -1 (b) 1 (c) 2 (d) 0

3 The point A divides the join of $P \equiv (-5, 1)$ and $Q \equiv (3, 5)$ in the ratio $k:1$. The two values of k for which the area of ΔABC, where $B \equiv (1, 5)$, $C \equiv (7, -2)$ is equal to 2 sq units are

(a) $\left(7, \frac{30}{9}\right)$ (b) $\left(7, \frac{31}{9}\right)$ (c) $\left(4, \frac{31}{9}\right)$ (d) $\left(7, \frac{31}{3}\right)$

4 If Δ_1 is the area of the triangle with vertices $(0, 0)$, $(a \tan\alpha, b \cot\alpha)$, $(a \sin\alpha, b\cos\alpha)$, Δ_2 is the area of the triangle with vertices (a, b), $(a \sec^2\alpha, b\,\text{cosec}^2\alpha)$, $(a + a\sin^2\alpha, b + b\cos^2\alpha)$ and Δ_3 is the area of the triangle with vertices $(0, 0)$, $(a\tan\alpha, -b\cot\alpha)$, $(a\sin\alpha, b\cos\alpha)$. Then,

(a) $\Delta_1, \Delta_2, \Delta_3$ are in GP (b) $\Delta_1, \Delta_2, \Delta_3$ are not in GP
(c) Cannot be discussed (d) None of these

5 If the area of the triangle formed by the points $O(0, 0)$, $A(a^{x^2}, 0)$ and $B(0, a^{6x})$ is $\frac{1}{2a^5}$ sq units, then $x =$

(a) 1, 5 (b) $-1, 5$ (c) $1, -5$ (d) $-1, -5$

6 The value of k for which the distinct points $(k, 2-2k)$, $(1-k, 2k)$ and $(-4-k, 6-2k)$ are collinear is (are)

(a) -1 or $1/2$ (b) Only $1/2$
(c) Only -1 (d) can not be found

7 If the line $2x + y = k$ passes through the point which divides the line segment joining the points (1, 1) and (2, 4) in the ratio 3 : 2, then k is equal to

(a) $\frac{29}{5}$ (b) 5 (c) 6 (d) $\frac{11}{5}$

8 A line L intersects the three sides BC, CA and AB of a ΔABC at P, Q and R, respectively. Then, $\frac{BP}{PC} \cdot \frac{CQ}{QA} \cdot \frac{AR}{RB}$ is equal to

(a) 1 (b) 0
(c) -1 (d) None of these

9 If the coordinates of the vertices of a triangle are integers, then the triangle cannot be

(a) equilateral (b) isosceles
(c) scalene (d) None of these

10 Let $O(0, 0)$, $P(3, 4)$ and $Q(6, 0)$ be the vertices of the ΔOPQ. The point R inside the ΔOPQ is such that ΔOPR, ΔPQR and ΔOQR are of equal area. Then, R is equal to

(a) $\left(\frac{4}{3}, 3\right)$ (b) $\left(3, \frac{2}{3}\right)$
(c) $\left(3, \frac{4}{3}\right)$ (d) $\frac{4}{3}$

11 The number of points having both coordinates as integers that lie in the interior of the triangle with vertices (0, 0), (0, 41) and (41, 0) is → **JEE Mains 2015**

(a) 901 (b) 861 (c) 820 (d) 780

12 If (0, 0), (1,1) and (1,0) be the middle points of the sides of a triangle, its incentre is

(a) $(2+\sqrt{2}, 2+\sqrt{2})$ (b) $[(2+\sqrt{2}, -(2+\sqrt{2})]$
(c) $(2-\sqrt{2}, 2-\sqrt{2})$ (d) $[(2-\sqrt{2}, -(2-\sqrt{2})]$

13 Vertices of a triangle are (1,2), (2,3) and (3,1) Its circumcentre is

(a) (11/6, 13/6) (b) (11/6, 2)
(c) (13/6, 11/6) (d) None of these

14 If a vertex of a triangle be (1,1) and the middle points of two sides through it be $(-2, 3)$ and $(5, 2)$, then the centroid of the triangle is

(a) (3, 5/3) (b) (3, 5)
(c) (5/3, 3) (d) None of these

15 The centroid of the triangle is (3,3) and the orthocentre is $(-3, 5)$ then its circumcentre is

(a) (0, 4) (b) (0, 8) (c) (6, 2) (d) $(6, -2)$

16 Let the orthocentre and centroid of a triangle be $A(-3, 5)$ and $B(3, 3)$ respectively. If C is the circumcentre of this triangle, then the radius of the circle having line segment AC as diameter, is → **JEE Mains 2018**

(a) $\sqrt{10}$ (b) $2\sqrt{10}$ (c) $3\sqrt{\frac{5}{2}}$ (d) $\frac{3\sqrt{5}}{2}$

17 If G is the centroid of ΔABC with vertices $A(a, 0)$, $B(-a, 0)$ and $C(b, c)$, then $\frac{(AB^2 + BC^2 + CA^2)}{(GA^2 + GB^2 + GC^2)}$ is equal to

(a) 1 (b) 2 (c) 3 (d) 4

18 Let a, b, c and d be non-zero numbers. If the point of intersection of the lines $4ax + 2ay + c = 0$ and $5bx + 2by + d = 0$ lies in the fourth quadrant and is equidistant from the two axes, then → **JEE Mains 2014**

(a) $2bc - 3ad = 0$ (b) $2bc + 3ad = 0$
(c) $2ad - 3bc = 0$ (d) $3bc - 2ad = 0$

19 The origin is shifted to (1,2). The equation $y^2 - 8x - 4y + 12 = 0$ changes to $y^2 = 4ax$, then a is equal to

(a) 1 (b) 2 (c) –2 (d) –1

20 If the axes are rotated through an angle of 60°, the coordinates of a point in the new system are $(2, -\sqrt{3})$, then its original coordinates are

(a) $\left(\frac{5}{3}, -\frac{\sqrt{2}}{3}\right)$ (b) $\left(-\frac{5}{3}, \frac{\sqrt{2}}{3}\right)$

(c) $\left(\frac{5}{2}, \frac{\sqrt{3}}{2}\right)$ (d) $\left(-\frac{5}{2}, -\frac{\sqrt{3}}{2}\right)$

21 By rotating the axes through 180°, the equation $x - 2y + 3 = 0$ changes to

(a) $x + 2y - 3 = 0$ (b) $x - 2y + 3 = 0$

(c) $x - 2y - 3 = 0$ (d) None of these

22 Let $0 < \alpha < \frac{\pi}{2}$ be a fixed angle. If $P = (\cos\theta, \sin\theta)$ and $Q = \{\cos(\alpha - \theta), \sin(\alpha - \theta)\}$, then Q is obtained from P by

(a) clockwise rotation around the origin through angles α

(b) anti-clockwise rotation around origin through angle α

(c) reflection in the line through the origin with slope $\tan\alpha$

(d) reflection in the line through the origin with slope $\tan\alpha/2$

23 The point (4, 1) undergoes the following transformations

(i) Reflection in the line $x - y = 0$

(ii) Translation through a distance of 2 units along positive direction of X-axis.

(iii) Projection on X-axis.

The coordinate of the point in its final position is

(a) (3, 4) (b) (3, 0) (c) (1, 0) (d) (4, 3)

24 If the points are $A(0, 4)$ and $B(0, -4)$, then find the locus of $P(x, y)$ such that $|AP - BP| = 6$.

(a) $9x^2 - 7y^2 + 63 = 0$ (b) $9x^2 + 7y^2 - 63 = 0$

(c) $9x^2 + 7y^2 + 63 = 0$ (d) None of these

25 ABC is a variable triangle with the fixed vertex $C(1, 2)$ and A, B having the coordinates $(\cos t, \sin t)$, $(\sin t, -\cos t)$ respectively, where t is a parameter. The locus of the centroid of the ΔABC is

(a) $3(x^2 + y^2) - 2x - 4y - 1 = 0$

(b) $3(x^2 + y^2) - 2x - 4y + 1 = 0$

(c) $3(x^2 + y^2) + 2x + 4y - 1 = 0$

(d) $3(x^2 + y^2) + 2x + 4y + 1 = 0$

26 If $A(2, -3)$ and $B(-2, 1)$ are two vertices of a triangle and third vertex moves on the line $2x + 3y = 9$, then the locus of the centroid of the triangle is

(a) $2x - 3y = 1$ (b) $x - y = 1$

(c) $2x + 3y = 1$ (d) $2x + 3y = 3$

27 Let $A(-3, 2)$ and $B(-2, 1)$ be the vertices of a ΔABC. If the centroid of this triangle lies on the line $3x + 4y + 2 = 0$, then the vertex C lies on the line

➔ **JEE Mains 2013**

(a) $4x + 3y + 5 = 0$ (b) $3x + 4y + 3 = 0$

(c) $4x + 3y + 3 = 0$ (d) $3x + 4y + 5 = 0$

28 The coordinates of points A and B are $(ak, 0)$ and $\left(\frac{a}{k}, 0\right)$, where $(k \neq \pm 1)$ if p moves in such a way that $PA = kPB$, the locus of P is

(a) $k^2(x^2 + y^2) = a^2$ (b) $x^2 + y^2 = k^2a^2$

(c) $x^2 + y^2 + a^2 = 0$ (d) $x^2 + y^2 = a^2$

29 If $A(-a, 0)$ and $B(a, 0)$ are two fixed points, then the locus of the point at which AB subtends a right angle is

(a) $x^2 + y^2 = 2a^2$ (b) $x^2 - y^2 = a^2$

(c) $x^2 + y^2 + a^2 = 0$ (d) $x^2 + y^2 = a^2$

30 A point moves in such a way that the sum of its distances from two fixed points $(ae, 0)$ and $(-ae, 0)$ is $2a$. Then the locus of the points is

(a) $\frac{x^2}{a^2} + \frac{y^2}{a^2(1 - e^2)} = 1$

(b) $\frac{x^2}{a^2} - \frac{y^2}{a^2(1 - e^2)} = 1$

(c) $\frac{x^2}{a^2(1 - e^2)} + \frac{y^2}{a^2} = 1$

(d) None of the above

Directions (Q. Nos. 31-35) *Each of these questions contains two statements : Statement* I *and Statement* II. *Each of these questions also has four alternative choices, only one of which is the correct answer. You have to select one of the codes* (*a*), (*b*), (*c*) *and* (*d*) *given below.*

(a) Statement I is true, Statement II is true; Statement II is a correct explanation for Statement I

(b) Statement I is true, Statement II is true; Statement II is not a correct explanation for Statement I

(c) Statement I is true; Statement II is false

(d) Statement I is false; Statement II is true

31 **Statement I** If $A(2a, 4a)$ and $B(2a, 6a)$ are two vertices of a equilateral ΔABC and the vertex C is given by $(2a + a\sqrt{3}, 5a)$.

Statement II An equilateral triangle all the coordinates of three vertices can be rational.

32 **Statement I** If the circumcentre of a triangle lies at the origin and centroid is the middle point of the line joining the points (2, 3) and (4, 7), then its orthocentre lies on the line $5x - 3y = 0$.

Statement II The circumcentre, centroid and the orthocentre of a triangle lie on the same line.

33 **Statement I** If the origin is shifted to the centroid of the triangle with vertices (0, 0), (3, 3) and (3, 6) without rotation of axes, then the vertices of the triangle in the new system of coordinates are (–2, 0), (1, 3) and (1, – 3).

Statement II If the origin is shifted to the point (2, 3) without rotation of the axes, then the coordinates of the point $P(\alpha - 1, \alpha + 1)$ in the new system of coordinates are $(\alpha - 3, \alpha - 2)$.

34 Let the equation of the line $ax + by + c = 0$.

Statement I If a, b and c are in AP, then $ax + by + c = 0$ pass through a fixed point $(1, -2)$.

Statement II Any family of lines always pass through a fixed point.

35 The lines $L_1 : y - x = 0$ and $L_2 : 2x + y = 0$ intersect the line $L_3 : y + 2 = 0$ at P and Q, respectively.

The bisector of the acute angle between L_1 and L_2 intersects L_3 at R.

Statement I The ratio $PR : RQ$ equals $2\sqrt{2} : \sqrt{5}$.

Statement II In any triangle, bisector of an angle divides the triangle into two similar triangles.

DAY PRACTICE SESSION 2

PROGRESSIVE QUESTIONS EXERCISE

1 If the coordinates of two points A and B are $(3, 4)$ and $(5, -2)$, respectively. Then, the coordinates of any point P, if $PA = PB$ and area of $\Delta PAB = 10$, are

(a) (7,5), (1, 0) (b) (7, 2), (1, 0)
(c) (7, 2), (−1, 0), (d) None of these

2 Lat $A(a, b)$ be a fixed point and O be the origin an coordinates. If A_1 is the mid-point at OA, A_2 is the mid-point at AA_1, A_3 is the mid-point at AA_2 and so on. Then, the coordinates of A_n are

(a) $(a(1-2^{-n}), b(1-2^{-n}))$ (b) $(a(2^{-n}-1), b(2^{-n}-1))$
(c) $(a(1-2^{-(n-1)}), b(1-2^{-(n-1)}))$ (d) None of these

3 The coordinates of points A, B, C are $(x_1, y_1), (x_2, y_2)$ and (x_3, y_3) and point D divides AB in the ratio $l : k$. If P divides line DC in the ratio $m : (k + l)$, coordinates of P are

(a) $\left(\dfrac{kx_1 + lx_2 + mx_3}{k + l + m}, \dfrac{ky_1 + ly_2 + my_3}{k + l + m}\right)$
(b) $\left(\dfrac{lx_1 + mx_2 + kx_3}{l + m + k}, \dfrac{ly_1 + my_2 + ky_3}{l + m + k}\right)$
(c) $\left(\dfrac{mx_1 + kx_2 + lx_3}{m + k + l}, \dfrac{my_1 + ky_2 + ly_3}{m + K + l}\right)$
(d) None of the above

4 The locus of a point P which moves such that $2PA = 3PB$, where $A(0, 0)$ and $B(4, -3)$ are points, is

(a) $5x^2 - 5y^2 - 72x + 54y + 225 = 0$
(b) $5x^2 + 5y^2 - 72x + 54y + 225 = 0$
(c) $5x^2 + 5y^2 + 72x - 54y + 225 = 0$
(d) $5x^2 + 5y^2 - 72x - 54y - 225 = 0$

5 Two points $P(a, 0)$ and $Q(-a, 0)$ are given, R is a variable point on one side of the line PQ such that $\angle RPQ - \angle RQP$ is 2α, then

(a) locus of R is $x^2 - y^2 + 2xy\cot 2\alpha - a^2 = 0$
(b) locus of R is $x^2 + y^2 + 2xy\cot\alpha - a^2 = 0$
(c) locus of R is a hyperbola, if $\alpha = \pi/4$
(d) locus of R is a circle, if $\alpha = \pi/4$

6 If the axis be turned through an angle $\tan^{-1} 2$, then what does the equation $4xy - 3x^2 = a^2$ become?

(a) $X^2 - 4Y^2 = a^2$ (b) $X^2 + 4Y^2 = a^2$
(c) $X^2 + 4Y^2 = -a^2$ (d) None of these

7 The orthocentre of the triangle whose vertices are $\{at_1 t_2, a(t_1 + t_2)\}, \{at_2 t_3, a(t_2 + t_3)\}, \{at_3 t_1, a(t_3 + t_1)\}$ is

(a) $\{-a, a(t_1 + t_2 + t_3 + t_1 + t_2 t_3)\}$
(b) $\{-a, a(t_1 + t_2 + t_3 + t_1 t_2 t_3)\}$
(c) $\{-a, a(t_1 - t_2 - t_3 - t_1 t_2 t_3)\}$
(d) $\{-a, a(t_1 + t_2 - t_3 - t_1 t_2 t_3)\}$

8 ABC is an isosceles triangle of area $\dfrac{25}{6}$ sq unit if the coordinates of base are $B(1, 3)$ and $C(-2, 7)$, the coordinates of A are

(a) $(1, 6), \left(-\dfrac{11}{6}, \dfrac{5}{6}\right)$ (b) $\left(-\dfrac{1}{2}, 5\right), \left(4, \dfrac{5}{6}\right)$
(c) $\left(\dfrac{5}{6}, 6\right), \left(-\dfrac{11}{6}, 4\right)$ (d) $\left(5, \dfrac{5}{6}\right), \left(\dfrac{11}{6}, 4\right)$

9 If $A(6, -3), B(-3, 5), C(4, -2), P(\alpha, \beta)$, then the ratio of the areas of the triangles PBC and ABC is

(a) $|\alpha + \beta|$ (b) $|\alpha - \beta|$
(c) $|\alpha + \beta + 2|$ (d) $|\alpha + \beta - 2|$

10 If O be the origin and if $P_1(x_1, y_1)$ and $P_2(x_2, y_2)$ be two points, then $|OP_1| \cdot |OP_2| \cos(\angle P_1OP_2)$ is equal to

(a) $x_1y_2 + x_2y_1$ (b) $(x_1^2 + y_1^2)(x_2^2 + y_2^2)$
(c) $(x_1 - x_2)^2 + (y_1 - y_2)^2$ (d) $x_1x_2 + y_1y_2$

11 If points $(0, 0), (2, 2\sqrt{3})$ and (a, b) are vertices of an equilateral triangle, then (a, b) is equal to

(a) (0, − 4) (b) (0, 4) (c) (4, 0) (d) (− 4, 0)

12 If the equation of the locus of a point equidistant from the points (a_1, b_1) and (a_2, b_2) is $(a_1 - a_2)x + (b_1 - b_2)y + c = 0$, then the value of c is

(a) $a_1^2 - a_2^2 + b_1^2 - b_2^2$ (b) $\sqrt{a_1^2 + b_1^2 - a_2^2 - b_2^2}$
(c) $\dfrac{1}{2}(a_1^2 + a_2^2 + b_1^2 + b_2^2)$ (d) $\dfrac{1}{2}(a_2^2 + b_2^2 - a_1^2 - b_1^2)$

13 If $(2, 1), (5, 2)$ and $(3, 4)$ are vertices of a triangle, its circumcentre is

(a) $\left(\dfrac{13}{2}, \dfrac{9}{2}\right)$ (b) $\left(\dfrac{13}{4}, \dfrac{9}{4}\right)$
(c) $\left(\dfrac{9}{4}, \dfrac{13}{4}\right)$ (d) $\left(\dfrac{9}{2}, \dfrac{13}{2}\right)$

14 A point moves is such a way that the sum of squares of its distances from $A(2, 0)$ and $B(-2, 0)$ is always equal to the square of the distance between A and B, then the locus of point P is

(a) $x^2 + y^2 - 2 = 0$ (b) $x^2 + y^2 + 2 = 0$
(c) $x^2 + y^2 + 4 = 0$ (d) $x^2 + y^2 - 4 = 0$

15 The area of a triangle is 5 and its two vertices are $A(2, 1)$ and $B(3, -2)$. The third vertex lies on $y = x + 3$. Then, third vertex is

(a) $\left(\frac{7}{2}, \frac{13}{2}\right)$ (b) $\left(\frac{5}{2}, \frac{5}{2}\right)$
(c) $\left(-\frac{3}{2}, -\frac{3}{2}\right)$ (d) $(0, 0)$

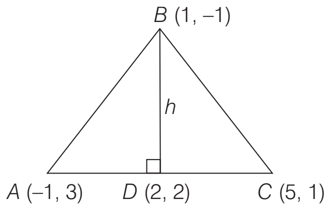
ANSWERS

SESSION 1	**1** (b)	**2** (c)	**3** (b)	**4** (b)	**5** (d)	**6** (c)	**7** (c)	**8** (c)	**9** (a)	**10** (c)
	11 (d)	**12** (c)	**13** (c)	**14** (c)	**15** (c)	**16** (c)	**17** (c)	**18** (c)	**19** (b)	**20** (c)
	21 (c)	**22** (d)	**23** (b)	**24** (a)	**25** (b)	**26** (c)	**27** (b)	**28** (d)	**29** (d)	**30** (a)
	31 (c)	**32** (a)	**33** (a)	**34** (a)	**35** (c)					
SESSION 2	**1** (b)	**2** (a)	**3** (a)	**4** (b)	**5** (a)	**6** (a)	**7** (b)	**8** (c)	**9** (d)	**10** (d)
	11 (c)	**12** (d)	**13** (b)	**14** (d)	**15** (a)					

Hints and Explanations

SESSION 1

1 Let BD be the median from B to AC, where D is the mid-point of AC.



According to mid-point formula coordinates of D are

$$\left(\frac{-1+5}{2}, \frac{3+1}{2}\right) = (2, 2)$$

$\therefore$ Length of median BD

$$= \sqrt{(2-1)^2 + (2+1)^2} = \sqrt{1+9} = \sqrt{10}$$

2 Condition of collinearity

$$\Delta = \begin{vmatrix} p+1 & 1 & 1 \\ 2p+1 & 3 & 1 \\ 2p+2 & 2p & 1 \end{vmatrix} = 0$$

$$\Rightarrow (p+1)(3-2p) - 1(2p+1-2p-2) + 1(4p^2 + 2p - 6p - 6) = 0$$

$$\Rightarrow -2p^2 + p + 3 + 1 + 4p^2 - 4p - 6 = 0$$

$$\Rightarrow \quad 2p^2 - 3p - 2 = 0$$

$$\Rightarrow 2p^2 - 4p + p - 2 = 0$$

$$\therefore \quad p = 2, -\frac{1}{2}$$

3 Coordinates of A, dividing the join of $P \equiv (-5, 1)$ and $Q \equiv (3, 5)$ in the ratio $k : 1$ are given by $A\left(\frac{3k-5}{k+1}, \frac{5k+1}{k+1}\right)$.

Also, area of ΔABC is given by

$$\Delta = \left|\frac{1}{2}\Sigma x_1 (y_2 - y_3)\right|$$

$$= \frac{1}{2}|[x_1(y_2 - y_3) + x_2(y_3 - y_1) + x_3(y_1 - y_2)]|$$

$$\Rightarrow \left|\frac{1}{2}\left\{\frac{3k-5}{k+1}(7) + 1\left(-2 - \frac{5k+1}{k+1}\right) + 7\left(\frac{5k+1}{k+1} - 5\right)\right\}\right| = 2$$

$$\Rightarrow \frac{1}{2}\left\{\frac{3k-5}{k+1}(7) + \left(-2 - \frac{5k+1}{k+1}\right) + 7\left(\frac{5k+1}{k+1} - 5\right)\right\} = \pm 2$$

$$\Rightarrow \quad 14k - 66 = 4k + 4$$

$$\Rightarrow \quad 10k = 70 \Rightarrow k = 7$$

$$\text{or} \quad 14k - 66 = -4k - 4$$

$$\Rightarrow \quad 18k = 62$$

$$\Rightarrow \quad k = \left(\frac{31}{9}\right)$$

Therefore, the values of k are 7 and $\frac{31}{9}$.

4 We have, $\Delta_1 = \frac{1}{2}\begin{Vmatrix} 0 & 0 & 1 \\ a\tan\alpha & b\cot\alpha & 1 \\ a\sin\alpha & b\cos\alpha & 1 \end{Vmatrix}$

$$= \frac{1}{2}ab|\sin\alpha - \cos\alpha|$$

and $\Delta_2 = \frac{1}{2}\begin{Vmatrix} a & b & 1 \\ a\sec^2\alpha & b\,\text{cosec}^2\alpha & 1 \\ a + a\sin^2\alpha & b + b\cos^2\alpha & 1 \end{Vmatrix}$

On applying $C_1 \to C_1 - aC_3$ and $C_2 \to C_2 - bC_3$, we get

$$\Delta_2 = \frac{1}{2}ab\begin{Vmatrix} 0 & 0 & 1 \\ \tan^2\alpha & \cot^2\alpha & 1 \\ \sin^2\alpha & \cos^2\alpha & 1 \end{Vmatrix}$$

$$= \frac{1}{2}ab\,|\sin^2\alpha - \cos^2\alpha|$$

and $\Delta_3 = \frac{1}{2}\begin{Vmatrix} 0 & 0 & 1 \\ a\tan\alpha & -b\cot\alpha & 1 \\ a\sin\alpha & b\cos\alpha & 1 \end{Vmatrix}$

$$= \frac{1}{2}ab\,|\sin\alpha + \cos\alpha|$$

So that, $\Delta_1\Delta_3 = \frac{1}{2}ab\Delta_2$

Suppose, Δ_1, Δ_2 and Δ_3 are in GP.

Then, $\Delta_1\Delta_3 = \Delta_2^2 \Rightarrow \frac{1}{2}ab\Delta_2 = \Delta_2^2$

$$\Rightarrow \quad \Delta_2 = \frac{1}{2}ab$$

$$\Rightarrow \frac{1}{2}ab(\sin^2\alpha - \cos^2\alpha) = \frac{1}{2}ab$$

$$\Rightarrow \quad \sin^2\alpha - \cos^2\alpha = 1$$

i.e. $\alpha = (2m+1)\frac{\pi}{2}, m \in I.$

But for this value of α, the vertices of the given triangles are not defined.

Hence, Δ_1, Δ_2 and Δ_3 cannot be in GP for any value of α.

5 We have,

area of $\Delta OAB = \frac{1}{2a^5}$ sq units

$$\Rightarrow \frac{1}{2} \times a^{x^2} \times a^{6x} = \frac{1}{2}a^{-5}$$

$$\Rightarrow a^{x^2+6x} = a^{-5}$$

$$\Rightarrow x^2 + 6x + 5 = 0$$

$$\Rightarrow x = -1, -5$$

6 Points are collinear so

$$\begin{vmatrix} k & 2-2k & 1 \\ 1-k & 2k & 1 \\ -4-k & 6-2k & 1 \end{vmatrix} = 0$$

$$\Rightarrow \begin{vmatrix} k & 2-2k & 1 \\ 1-2k & 4k-2 & 0 \\ -4-2k & 4 & 0 \end{vmatrix} = 0$$

[applying $C_2 \to C_2 - C_1$ and $C_3 \to C_3 - C_2$]

$$\Rightarrow 4 - 8k + (4k-2)(4+2k) = 0$$

$$\Rightarrow \quad 2k^2 + k - 1 = 0$$

so $k = -1$ and $1/2$

But for $k = \frac{1}{2}$, points are $(1/2, 1)$, $(1/2, 1)$ and $\left(-\frac{9}{5}, 5\right)$

Which is a contradiction as given points are distinct.

7 Using section formula, the coordi- nates of the point P, which divides AB internally in the ratio 3 : 2 are

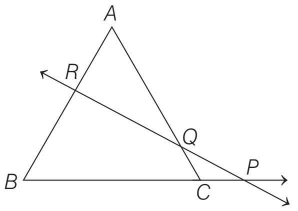
$$P\left(\frac{3\times2+2\times1}{3+2}, \frac{3\times4+2\times1}{3+2}\right)$$

$$\equiv P\left(\frac{8}{5}, \frac{14}{5}\right)$$

Also, since the line L passes through P, hence substituting the coordinates of $P\left(\frac{8}{5}, \frac{14}{5}\right)$ in the equation of $L: 2x + y = k$, we get

$$2\left(\frac{8}{5}\right) + \left(\frac{14}{5}\right) = k \Rightarrow k = 6$$

8 Let $A(x_1, y_1)$, $B(x_2, y_2)$ and $C(x_3, y_3)$ be the vertices of ΔABC and $lx + my + n = 0$ be the equation of the line. If P divides BC in the ratio λ:1, then the coordinates of P are $\left(\frac{\lambda x_3 + x_2}{\lambda+1}, \frac{\lambda y_3 + y_2}{\lambda+1}\right)$.



Also, as P lies on L, we have

$$l\left(\frac{\lambda x_3 + x_2}{\lambda+1}\right) + m\left(\frac{\lambda y_3 + y_2}{\lambda+1}\right) + n = 0$$

$$\Rightarrow \quad -\frac{lx_2 + my_2 + n}{lx_3 + my_3 + n} = \frac{BP}{PC} = \lambda \quad \text{...(i)}$$

Similarly, we obtain

$$\frac{CQ}{QA} = -\frac{lx_3 + my_3 + n}{lx_1 + my_1 + n} \quad \text{...(ii)}$$

and $$\frac{AR}{RB} = -\frac{lx_1 + my_1 + n}{lx_2 + my_2 + n} \quad \text{...(iii)}$$

On multiplying Eqs. (i), (ii) and (iii), we get

$$\frac{BP}{PC} \cdot \frac{CQ}{QA} \cdot \frac{AR}{PB} = -1$$

9 Let $A = (x_1, y_1)$, $B = (x_2, y_2)$, $C = (x_3, y_3)$ be the vertices of a triangle and x_1, x_2, x_3 and y_1, y_2, y_3 be integers.

So, $BC^2 = (x_2 - x_3)^2 + (y_2 - y_3)^2$ is a positive integers.

If the triangle is equilateral, then

$$AB = BC = CA = a \quad \text{[say]}$$

and $$\angle A = \angle B = \angle C = 60°.$$

$\therefore$ Area of the triangle $= \left(\frac{1}{2}\right)\sin A \cdot bc$

$$= \left(\frac{1}{2}\right)a^2 \sin 60°$$

$$= \left(\frac{a^2}{2}\right)\left(\frac{\sqrt{3}}{2}\right)$$

$$= \frac{\sqrt{3}}{4}a^2$$

which is irrational.

[since, a^2 is a positive integer]

Now, the area of the triangle in terms of the coordinates

$$= \frac{1}{2}[x_1(y_2 - y_3) + x_2(y_3 - y_1) + x_3(y_1 - y_2)]$$

which is a rational number.

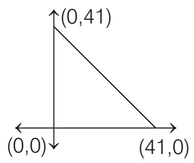
This contradicts that the area is an irrational number, if the triangle is equilateral.

10 If the centroid is joined to the vertices, we get three triangles of equal area.

$$\therefore \quad R = G = \left(3, \frac{4}{3}\right)$$

11 Required points (x, y) are such that, it satisfy $x + y < 41$ and $x > 0$, $y > 0$.

Number of positive integral solution of the equation $x + y + k = 41$ will be number of integral coordinates in the bounded region.



$\therefore$ Total number of integral coordinates

$$= {}^{41-1}C_{3-1} = {}^{40}C_2$$

$$= \frac{40!}{2!\,38!} = 780$$

12 If $A(x_1, y_1)$, $B(x_2, y_2)$, $C(x_3, y_3)$ be the vertices of the triangle and if (0, 0), (1, 1) and (1,0) are the middle points of AB, BC and CA respectively, then

$$x_1 + x_2 = 0,\ x_2 + x_3 = 2,\ x_3 + x_1 = 2$$

$$y_1 + y_2 = 2,\ y_2 + y_3 = 2,\ y_3 + y_1 = 0$$

So, $A(0, 0)$, $B(0, 2)$ and $C(2, 0)$ are the vertices of the ΔABC.

Now, $a = BC = 2\sqrt{2}$, $b = CA = 2$, $c = AB = 2$

The coordinates (α, β) of the in-centre are given by

$$\alpha = \frac{ax_1 + bx_2 + cx_3}{a+b+c} = 2 - \sqrt{2},$$

$$\beta = \frac{ay_1 + by_2 + cy_3}{a+b+c} = 2 - \sqrt{2}$$

i.e. The in-centre is $(2 - \sqrt{2}, 2 - \sqrt{2})$.

13 $(x-1)^2 + (y-2)^2$

$$= (x-2)^2 + (y-3)^2$$

$$= (x-3)^2 + (y-1)^2$$

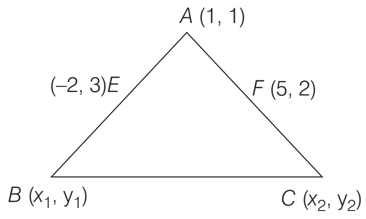
$$\Rightarrow x + y = 4,\ 4x - 2y = 5$$

$$\Rightarrow \quad x = 13/6,\ y = 11/6$$

$\therefore$ Circumcentre $= \left(\frac{13}{6}, \frac{11}{6}\right)$

14 $\frac{x_1 + 1}{2} = -2$

$$\Rightarrow \quad x_1 = -5$$



$$\frac{y_1 + 1}{2} = 3 \Rightarrow y_1 = 5$$

$$B = (-5, 5)$$

$$\frac{x_2 + 1}{2} = 5 \Rightarrow x_2 = 9$$

$$\frac{y_2 + 1}{2} = 2 \Rightarrow y_2 = 3$$

$$\therefore \quad C = (9, 3)$$

$$G = \left(\frac{1 + 9 - 5}{3}, \frac{1 + 5 + 3}{3}\right)$$

$$= (5/3, 3)$$

15 Since, we know that

Centroid divides the join of orthocenter and circumcenter in the ratio of 2 : 1

Let the circumcenter of Δ is (α, β)

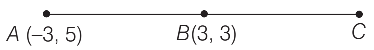
$\Rightarrow G\,(3, 3) = G\left(\frac{2\alpha - 3}{3}, \frac{2\beta + 5}{3}\right)$

$\therefore \frac{2\alpha - 3}{3} = 3$ and $\frac{2\beta + 5}{3} = 3$

or $\alpha = 6, \beta = 2$

$\therefore$ Circumcentre of Δ is $C(6, 2)$.

16 We have, orthocentre and centroid of a triangle be $A(-3, 5)$ and $B(3, 3)$ respectively and C circumcentre



We know that,

$AB : BC = 2 : 1$

$AB = \sqrt{(3+3)^2 + (3-5)^2}$

$= \sqrt{36 + 4} = 2\sqrt{10}$

$\therefore \quad BC = \sqrt{10}$

$AC = AB + BC$

$= 2\sqrt{10} + \sqrt{10} = 3\sqrt{10}$

Since, AC is a diametre of circle

$\therefore \quad r = \frac{AC}{2} = \frac{3\sqrt{10}}{2} = 3\sqrt{\frac{5}{2}}$

17 Coordinates of point G is $G\left(\frac{b}{3}, \frac{c}{3}\right)$

Let $E = \frac{(AB)^2 + (BC)^2 + (CA)^2}{(GA)^2 + (GB)^2 + (GC)^2}$

$$\Rightarrow E = \frac{4a^2 + (a+b)^2 + c^2 + (a-b)^2 + c^2}{\left(\frac{b}{3} - a\right)^2 + \left(\frac{c}{3}\right)^2 + \left(\frac{b}{3} + a\right)^2 + \left(\frac{c}{3}\right)^2 + \left(\frac{2b}{3}\right)^2 + \left(\frac{2c}{3}\right)^2}$$

$$\Rightarrow E = \frac{4a^2 + 2c^2 + 2a^2 + 2b^2}{\frac{2b^2}{9} + 2a^2 + \frac{6c^2}{9} + \frac{4b^2}{9}}$$

$$\Rightarrow E = \frac{6a^2 + 2b^2 + 2c^2}{\frac{1}{9}(6b^2 + 18a^2 + 6c^2)}$$

$$\Rightarrow E = \frac{2(3a^2 + b^2 + c^2)}{\frac{1}{9}6(3a^2 + b^2 + c^2)} = 3$$

18 Let coordinate of the intersection point in fourth quadrant be $(\alpha, -\alpha)$.

Since, $(\alpha, -\alpha)$ lies on both lines

$4ax + 2ay + c = 0$ and

$5bx + 2by + d = 0$.

$\therefore \quad 4a\alpha - 2a\alpha + c = 0$

$\Rightarrow \quad \alpha = \frac{-c}{2a}$ ...(i)

and $5b\alpha - 2b\alpha + d = 0$

$\Rightarrow \quad \alpha = \frac{-d}{3b}$ ...(ii)

From Eqs. (i) and (ii), we get

$\frac{-c}{2a} = \frac{-d}{3b} \Rightarrow 3bc = 2ad$

$\Rightarrow \quad 2ad - 3bc = 0$ ...(iii)

19 Let $P(x, y)$ be the original position of the point w.r.t the original axes. Let us move the origin at new position to (h, k). Hence, the position of the same point P in the new system is

$x' = x - h$

$y' = y - k$

Here, $(h, k) = (1, 2)$

$\therefore \quad x' = (x - 1)$

$y' = (y - 2)$

As per the given situation

$y^2 - 8x - 4y + 12 = (y-2)^2 - 4a(x-1)$

$\Rightarrow y^2 - 8x - 4y + 12 = y^2 - 4y + 4 - 4ax + 4a$

Comparing respective coefficients, we have

$4a = 8$

$\therefore \quad a = 2$

20 Let $P(a', y')$ be the coordinates of the point obtained by rotating the axes through an angle of 60°.

$\therefore$ The transformation matrix can be written as

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 2 \\ -\sqrt{3} \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & \frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & \frac{1}{2} \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 2 \\ -\sqrt{3} \end{bmatrix} = \begin{bmatrix} \frac{x}{2} + \frac{\sqrt{3}y}{2} \\ -\frac{\sqrt{3}x}{2} + \frac{y}{2} \end{bmatrix}$$

$\Rightarrow \quad x + \sqrt{3}y = 4$ and $\sqrt{3}x - y = 2\sqrt{3}$

Solving the above equations,

we have $(x, y) = \left(\frac{5}{2}, \frac{\sqrt{3}}{2}\right)$

21 Let $P(x', y')$ be the coordinates of the point $P(x, y)$ after rotation of axes at an angle of 180°

$$\Rightarrow \begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

Since, here $\theta = 180°$

$$\Rightarrow \begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} -x \\ -y \end{bmatrix}$$

$\therefore \quad x = -x'$ and $y = -y'$

Hence, the new equation of curve,

$x - 2y + 3 = 0$ is $(-x') - 2(-y') + 3 = 0$

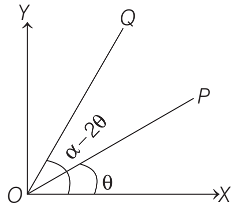
$\Rightarrow -x' + 2y' + 3 = 0$

$\Rightarrow \quad x' - 2y' - 3 = 0$

or $\quad x - 2y - 3 = 0$ in general

22 OP is inclined at angle θ with X-axis OQ is inclined at angle $\alpha - 2\theta$ with X-axis. The bisector of angle POQ is inclined at angle

$\frac{\alpha - 2\theta}{2} + \theta = \frac{\alpha}{2}$ with X-axis.



23 Image of (4, 1) in the line $x = y$ is (1, 4) on translating this point along positive direction of X-axis by 2 units, this point is transformed into (3, 4) and projection of the point (3, 4) on X-axis is (3, 0).

24 $BP - AP = \pm 6$ or $BP = AP \pm 6$

$\Rightarrow \sqrt{x^2 + (y+4)^2} = \sqrt{x^2 + (y-4)^2} \pm 6$

On squaring and simplifying, we get

$4y - 9 = \pm 3\sqrt{x^2 + (y-4)^2}$

Again on squaring, we get

$9x^2 - 7y^2 + 63 = 0$

25 Let $G(\alpha, \beta)$ be the centroid in any position. Then,

$(\alpha, \beta) = \left(\frac{1 + \cos t + \sin t}{3}, \frac{2 + \sin t - \cos t}{3}\right)$

$\therefore \quad \alpha = \frac{1 + \cos t + \sin t}{3}$

and $\beta = \frac{2 + \sin t - \cos t}{3}$

$\Rightarrow \quad 3\alpha - 1 = \cos t + \sin t$ ...(i)

and $3\beta - 2 = \sin t - \cos t$ ...(ii)

On squaring and adding Eqs. (i) and (ii), we get

$(3\alpha - 1)^2 + (3\beta - 2)^2 = (\cos t + \sin t)^2 + (\sin t - \cos t)^2$

$= 2(\cos^2 t + \sin^2 t) = 2$

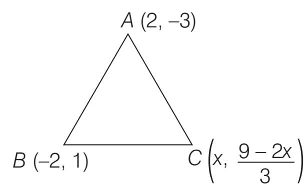
$\therefore$ The equation of the locus of the centroid is $(3x - 1)^2 + (3y - 2)^2 = 2$

$\Rightarrow \quad 9(x^2 + y^2) - 6x - 12y + 3 = 0$

$\Rightarrow \quad 3(x^2 + y^2) - 2x - 4y + 1 = 0$

26 The third vertex lies on $2x + 3y = 9$

i.e. $\left(x, \frac{9 - 2x}{3}\right)$



$\therefore$ Locus of centroid is

$$\left(\frac{2-2+x}{3}, \frac{-3+\frac{9-2x}{3}+1}{3}\right) = (h, k)$$

$$\therefore \quad h = \frac{x}{3} \text{ and } k = \frac{3-2x}{9}$$

$\Rightarrow \quad 9k = 3 - 2(3h) \Rightarrow 9k = 3 - 6h$

$\Rightarrow \quad 2h + 3k = 1$

Hence, locus of a point is $2x + 3y = 1$.

27 Let third vertex be $C(x_1, y_1)$.

$\therefore$ Centroid $\left(\frac{-3-2+x_1}{3}, \frac{2+1+y_1}{3}\right)$ lies on line

$$3x + 4y + 3 = 0$$

28 Let $P(\alpha, \beta)$ be any point such that

$$(PA) = k(PB)$$

$\Rightarrow \quad (PA)^2 = k^2(PB)^2$

$\Rightarrow \quad (\alpha - ak)^2 + \beta^2$

$$= k^2\left\{\left(\alpha - \frac{a}{k}\right)^2 + \beta^2\right\}$$

$\Rightarrow \alpha^2 + \beta^2 - 2ak\alpha + a^2k^2 = k^2\alpha^2$

$$+ k^2\beta^2 - \frac{2ak^2}{k}\alpha + a^2$$

$\Rightarrow (1 - k^2)\alpha^2 + (1 - k^2)\beta^2 = (1 - k^2)a^2$

$\Rightarrow (1 - k^2)\{\alpha^2 + \beta^2\} = (1 - k^2)a^2$

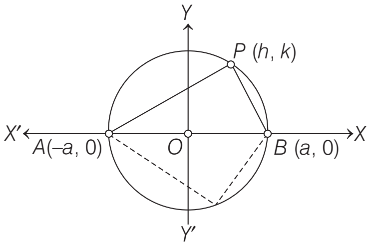
$\{\because k \neq \pm 1\}$

$\therefore \quad \alpha^2 + \beta^2 = a^2$

Replace α by x and β by y, we have

$$x^2 + y^2 = a^2$$

29 Let $P(h, k)$ represents all those points subtending a right angle at A and B



$\therefore \quad m_{AP} \cdot m_{PB} = -1$

$\Rightarrow \quad \left(\frac{k-0}{h+a}\right)\left(\frac{k-0}{h-a}\right) = -1$

$\Rightarrow \quad k^2 = -(h^2 - a^2)$

$\Rightarrow \quad k^2 + h^2 = a^2$

Replace $k \to y$ and $h \to x$, we get

$$x^2 + y^2 = a^2$$

30 Since, $A(ae, 0)$ and $B(-ae, 0)$ be the given points and let $P(h, k)$ be any point whose distance from A and B is constant i.e. $2a$.

i.e. $|PA| + |PB| = 2a$

$\Rightarrow \sqrt{(h - ae)^2 + k^2}$

$$+ \sqrt{(h + ae)^2 + k^2} = 2a \quad \ldots\text{(i)}$$

Let us assume

$\{(h - ae)^2 + k^2\} - \{(h + ae)^2 + k^2\}$

$$= -4aeh \quad \ldots\text{(ii)}$$

On dividing Eqs. (ii) by (i), we have

$$\frac{\{(h - ae)^2 + k^2\} - \{(h + ae)^2 + k^2\}}{\sqrt{(h - ae)^2 + k^2} + \sqrt{(h + ae)^2 + k^2}} = \frac{-4aeh}{2a}$$

$\Rightarrow \sqrt{(h - ae)^2 + k^2} - \sqrt{(h + ae)^2 + k^2}$

$$= -2eh \quad \ldots\text{(iii)}$$

$\{\because a - b = (\sqrt{a} + \sqrt{b})(\sqrt{a} - \sqrt{b})\}$

Adding Eqs. (i) and (iii), we have

$$2\sqrt{(h - ae)^2 + k^2} = (2a - 2eh)$$

$\Rightarrow \quad 2\sqrt{(h - ae)^2 + k^2} = 2(a - eh)$

Squaring both sides, we have

$$(h - ae)^2 + k^2 = (a - eh)^2$$

$\Rightarrow h^2 + a^2e^2 - 2aeh + k^2$

$$= a^2 + e^2h^2 - 2aeh$$

$\Rightarrow \quad h^2 - e^2h^2 + k^2 = a^2 - a^2e^2$

$\Rightarrow \quad h^2(1 - e^2) + k^2 = a^2(1 - e^2)$

Replacing h by x and k by y, we get the locus of point $P(h, k)$ which is the locus of an ellipse.

$$\frac{x^2}{a^2} + \frac{y^2}{a^2(1 - e^2)} = 1$$

31 **Statement I** : $AB = BC = CA$

$\therefore A, B, C$ are the vertices of triangle ABC.

Statement II : Let $A(x_1, y_1), B(x_2, y_2)$ and $C(x_3, y_3)$ are all rational coordinates.

$$\therefore \text{ Area } (\Delta ABC) = \frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} = \frac{\sqrt{3}}{4}$$

$$[(x_1 - x_2)^2 + (y_1 - y_2)^2]$$

LHS = rational, RHS = irrational

Hence, $(x_1, y_1), (x_2, y_2)$ and (x_3, y_3) cannot be all rational.

32 The orthocentre lies on the line joining the points (0, 0) and (3, 5) i.e. $5x - 3y = 0$.

Also, Statement II is true.

33 Statement II is true as the coordinates of the point P in new system are $(\alpha - 1 - 2, \alpha + 1 - 3)$.

In Statement I, the centroid is (2, 3), so the coordinates of the vertices in the new system of coordinates are $(-2, -3)$, $(1, 0)$, $(1, 3)$.

34 Statement II is false as $L_1 + \lambda L_2 = 0$

$\Rightarrow$ Family of concurrent lines, if L_1 and L_2 are intersect.

$\Rightarrow$ Family of parallel lines, if L_1 and L_2 are parallel.

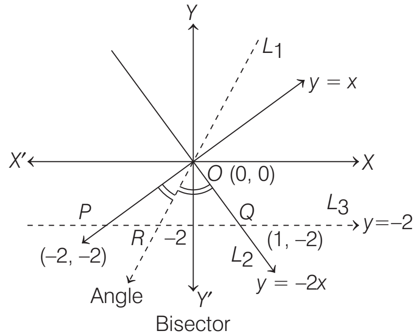
$\Rightarrow$ Family of coincident lines, if L_1 and L_2 are coincident.

As a, b and c are in AP.

$\Rightarrow \quad 2b = a + c \Rightarrow a - 2b + c = 0$

On comparing with $ax + by + c = 0$, it passes through fixed points $(1, -2)$.

35 Here, $L_1 : y - x = 0$ and $L_2 : 2x + y = 0$ and $L_3 : y + 2 = 0$ as shown below,



$|PO| = \sqrt{4 + 4}$

$$= 2\sqrt{2}; |OQ| = \sqrt{1 + 4} = \sqrt{5}$$

Since, OR is angle bisector

$$\frac{OP}{OQ} = \frac{PR}{RQ}$$

$$\Rightarrow \quad \frac{PR}{RQ} = \frac{2\sqrt{2}}{\sqrt{5}}$$

Hence, Statement I is true.

But, it does not divide the triangle in two similar triangles.

Hence, Statement II is false.

SESSION 2

1 Let the coordinates of P be (x, y).

Then, $PA = PB \Rightarrow PA^2 = PB^2$

$\Rightarrow (x - 3)^2 + (y - 4)^2 = (x - 5)^2 + (y + 2)^2$

$\Rightarrow \quad x - 3y - 1 = 0 \quad \ldots\text{(i)}$

Now, area of $\Delta PAB = 10$

$$\Rightarrow \quad \frac{1}{2}\begin{vmatrix} x & y & 1 \\ 3 & 4 & 1 \\ 5 & -2 & 1 \end{vmatrix} = \pm 10$$

$\Rightarrow \quad 6x + 2y - 26 = \pm 20$

$\Rightarrow \quad 6x + 2y - 46 = 0$

or $\quad 6x + 2y - 6 = 0$

$\Rightarrow \quad 3x + y - 23 = 0$

or $\quad 3x + y - 3 = 0 \quad \ldots\text{(ii)}$

On solving, $x - 3y - 1 = 0$ and $3x + y - 23 = 0$, we get

$$x = 7, y = 2$$

On solving $x - 3y - 1 = 0$ and $3x + y - 3 = 0$, we get

$$x = 1, y = 0$$

Thus, the coordinates of P are $(7, 2)$ or $(1, 0)$.

2 The coordinates of A_1 are $\left(\frac{a}{2},\frac{b}{2}\right)$

The coordinates of A_2 are $\left(\frac{a+\frac{a}{2}}{2},\frac{b+\frac{b}{2}}{2}\right)$

$=\left(\frac{a}{2}+\frac{a}{2^2},\frac{b}{2}+\frac{b}{2^2}\right)$

The coordinates of A_3 are

$=\left(\frac{a+\frac{a}{2}+\frac{a}{2^2}}{2},\frac{b+\frac{b}{b^2}+\frac{b}{2^2}}{2}\right)$

$=\left(\frac{a}{2}+\frac{a}{2^2}+\frac{a}{2^3},\frac{b}{2}+\frac{b}{2^2}+\frac{b}{2^3}\right)$

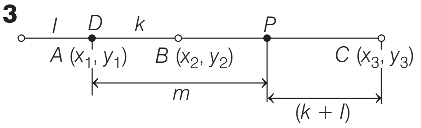
Continuing in this manner we observe that the coordinates of A_n are

$\left(\frac{a}{2}+\frac{a}{2^2}+\frac{a}{2^3}+\ldots+\frac{a}{2^n},\ \frac{b}{2}+\frac{b}{2^2}+\frac{b}{2^3}+\ldots+\frac{b}{2^n}\right)$

$=\left(a\left(1-\frac{1}{2^n}\right),b\left(1-\frac{1}{2^n}\right)\right)$

$=(a(1-2^{-n}),b(1-2^{-n}))$

3



Coordinates of point D are

$$D\left(\frac{lx_2+kx_1}{k+l},\frac{ly_2+ky_1}{k+l}\right)$$

Coordinates of point P are

$$P\left(\frac{lx_2+kx_1+mx_3}{k+l+m},\frac{ly_2+ky_1+my_3}{k+l+m}\right)$$

4 Let $P(h,k)$ be any point such that

$2(PA)=3(PB)$

$\Rightarrow 4(PA)^2=9(PB)^2$

$\Rightarrow 4(h^2+k^2)=9\{(h-4)^2+(k+3)^2\}$

$\Rightarrow 4(h^2+k^2)=9(h^2+k^2-8h+6k+25)$

$\Rightarrow 5h^2+5k^2-72h+54k+225=0$

$\therefore$ Required locus is

$5x^2+5y^2-72x+54y+225=0$

5 Let $\angle RPQ=\theta$ and $\angle RQP=\phi$

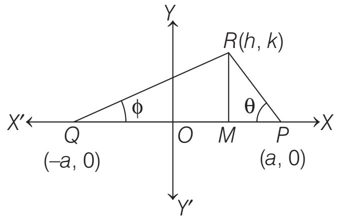
$\therefore \quad \theta-\phi=2\alpha$

Let $RM\perp PQ$, so that $RM=k$, $MP=a-h$

and $MQ=a+h$

Then, $\tan\theta=\frac{RM}{MP}=\frac{k}{a-h}$

and $\tan\phi=\frac{RM}{MQ}=\frac{k}{a+h}$



Again, now $2\alpha=\theta-\phi$

$\therefore \quad \tan 2\alpha=\tan(\theta-\phi)$

$=\frac{\tan\theta-\tan\phi}{1+\tan\theta\tan\phi}$

$=\frac{k(a+h)-k(a-h)}{a^2-h^2+k^2}$

$\Rightarrow a^2-h^2+k^2=2hk\cot 2\alpha$

Hence, the locus is

$x^2-y^2+2xy\cot 2\alpha-a^2=0$

6 Here, $\tan\theta=2$

So, $\cos\theta=\frac{1}{\sqrt5}$, $\sin\theta=\frac{2}{\sqrt5}$

For x and y, we have

$x=X\cos\theta-Y\sin\theta=\frac{X-2Y}{\sqrt5}$

and $y=X\sin\theta+Y\cos\theta=\frac{2X+Y}{\sqrt5}$

The equation $4xy-3x^2=a^2$ reduces to

$$\frac{4(X-2Y)}{\sqrt5}\cdot\frac{(2X+Y)}{\sqrt5}-3\left(\frac{X-2Y}{\sqrt5}\right)^2=a^2$$

$\Rightarrow 4(2X^2-2Y^2-3XY)-3(X^2-4XY+4Y^2)=5a^2$

$\Rightarrow 5X^2-20Y^2=5a^2$

$\therefore X^2-4Y^2=a^2$

7 Let the vertices be C, A and B, respectively. The altitude from A is

$\frac{y-a(t_2+t_3)}{x-at_2t_3}=-t_1$

$\Rightarrow xt_1+y=at_1t_2t_3+a(t_2+t_3)$ ...(i)

The altitude from B is

$xt_2+y=at_1t_2t_3+a(t_3+t_1)$ ...(ii)

On subtracting Eq. (ii) from Eq. (i), we get $x=-a$

Hence, $y=a(t_1+t_2+t_3+t_1t_2t_3)$

So, the orthocentre is $\{-a, a(t_1+t_2+t_3+t_1t_2t_3)\}$.

8 Given that, the triangle ABC is isosceles

$\therefore |AB|=|AC|$

Let the coordinate of A are $A(h,k)$

$\therefore \sqrt{(h-1)^2+(k-3)^2}=\sqrt{(h+2)^2+(k-7)^2}$

$\Rightarrow (h-1)^2+(k-3)^2=(h+2)^2+(k-7)^2$

$\Rightarrow -2h+1-6k+9=4h+4-14k+49$

$\Rightarrow 6h-8k+43=0$ ...(i)

Since, the area of triangle is 10 sq unit {given}

$\text{ar}(\Delta ABC)=\frac{1}{2}|BC||AC|$

$\Rightarrow \left|\frac{1}{2}(5)\sqrt{\left(h+\frac{1}{2}\right)^2+(k-5)^2}\right|=\frac{25}{6}$

On squaring, we get

$\Rightarrow \left(h+\frac{1}{2}\right)^2+(k-5)^2=\frac{25}{9}$

Using Eq. (i), we have

$\Rightarrow \left(\frac{8k-43}{6}+\frac{1}{2}\right)^2+(k-5)^2=\frac{25}{9}$

$\Rightarrow (4k-20)^2+9(k-5)^2=25$

$\Rightarrow 25\cdot(k-5)^2=25$

$\Rightarrow (k-5)^2=1$

$\Rightarrow |k-5|=1$

$\Rightarrow k-5=\pm1$

$\therefore k=1+5$ or $k=-1+5$

$\Rightarrow k=6$ or $k=4$

Using Eq. (i), we have $h=\frac{5}{6}$

Using Eq. (i), we have

$h=-\frac{11}{6}$

Therefore, the vertex A of the isosceles ΔABC is $A\left(\frac{5}{6},6\right)$ or $A\left(-\frac{11}{6},4\right)$.

9 $\text{ar}(\Delta PBC)=\frac{1}{2}\left\|\begin{matrix}\alpha & \beta & 1\\ -3 & 5 & 1\\ 4 & -2 & 1\end{matrix}\right\|$

$\Rightarrow \text{ar}(\Delta PBC)=\frac{1}{2}|7\alpha+7\beta-14|$

$=\frac{7}{2}|\alpha+\beta-2|$

Also, $\text{ar}(\Delta ABC)=\frac{1}{2}\left\|\begin{matrix}6 & -3 & 1\\ -3 & 5 & 1\\ 4 & -2 & 1\end{matrix}\right\|$

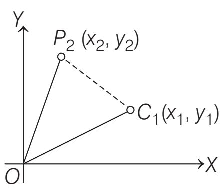
$\Rightarrow \text{ar}(\Delta ABC)=\frac{1}{2}|42-21-14|=\frac{7}{2}$

$\frac{\text{ar}(\Delta PBC)}{\text{ar}(\Delta ABC)}=|\alpha+\beta-2|$

10 By PROJECTION FORMULA, we have

$\cos\angle P_1OP_2=\frac{|OP_1|^2+|OP_2|^2-|P_1P_2|^2}{2|OP_1||OP_2|}$

Let $E=|OP_1||OP_2|\cos\angle P_1OP_2$



$$\Rightarrow E = \frac{(x_1^2 + y_1^2) + (x_2^2 + y_2^2) - \{(x_2 - x_1)^2 + (y_2 - y_1)^2\}}{2}$$

$$\Rightarrow E = \frac{[x_1^2 + x_2^2 + y_1^2 + y_2^2 - (x_2 - x_1)^2 - (y_2 - y_1)^2]}{2}$$

$$\Rightarrow E = \frac{2x_1x_2 + 2y_1y_2}{2}$$

$\therefore |OP_1||OP_2| \cos \angle P_1OP_2 = x_1x_2 + y_1y_2$

11 The points $A(0, 0)$, $B(2, 2\sqrt{3})$ and $C(a, b)$ are the vertices of an equilateral triangle if

$|AB| = |BC| = |CA|$

$\Rightarrow |AB|^2 = |BC|^2 = |CA|^2$

$\Rightarrow 4 + 12 = (a-2)^2 + (b - 2\sqrt{3})^2$

$= a^2 + b^2$

Now, $(a-2)^2 + (b - 2\sqrt{3})^2 = a^2 + b^2$

$a^2 + b^2 - 4a - 4\sqrt{3}b + 16 = a^2 + b^2$

$a + \sqrt{3}b = 4$

$a = 4 - \sqrt{3}b$ ...(i)

Also, $a^2 + b^2 = 16$

$(4 - \sqrt{3}b)^2 + b^2 = 16$ [using Eq. (i)]

$\Rightarrow 4b^2 - 8\sqrt{3}b + 16 = 16$

$\Rightarrow 4b(b - 2\sqrt{3}) = 0$

$\Rightarrow b = 0$ or $b = 2\sqrt{3}$

If $b = 0$

$\Rightarrow a = 4$

or if $b = 2\sqrt{3}$

$\Rightarrow a = -2$ [using Eq. (i)]

12 Let (h, k) be the point on the locus. Then by the given conditions

$(h - a_1)^2 + (k - b_1)^2$

$= (h - a_2)^2 + (k - b_2)^2$

$\Rightarrow 2h(a_1 - a_2) + 2k(b_1 - b_2) + a_2^2 + b_2^2 - a_1^2 - b_1^2 = 0$

$\Rightarrow h(a_1 - a_2) + k(b_1 - b_2) + \frac{1}{2}(a_2^2 + b_2^2 - a_1^2 - b_1^2) = 0$ ...(i)

Since, the locus of (h, k) is the line

$(a_1 - a_2)h + (b_1 - b_2)k + c = 0$ ...(ii)

$\therefore$ Comparing Eqs. (i) and (ii), we get

$c = \frac{1}{2}(a_2^2 + b_2^2 - a_1^2 - b_1^2)$

13 Circumcentre of a triangle is the point which is equidistant from the vertices of a triangle.

Let the circumcentre of triangle be $C(x, y)$ and the three vertices of the triangle are represented by $P(2, 1)$, $Q(5, 2)$, $R(3, 4)$

$\therefore$ According to given condition, we have

$|PC| = |QC| = |RC|$

Case I $|PC| = |QC|$

$(x-2)^2 + (y-1)^2 = (x-5)^2 + (y-2)^2$

$\Rightarrow 6x + 2y = 24$ ...(i)

Case II $|PC| = |RC|$

$(x-2)^2 + (y-1)^2 = (x-3)^2 + (y-4)^2$

$\Rightarrow 2x + 6y = 20$ ...(ii)

Solving Eqs. (i) and (ii) for x and y, we have

$\therefore$ Co-ordinates of circumcentre are

$C(x, y) = C\left(\frac{13}{4}, \frac{9}{4}\right)$

14 Let $P(h, k)$ be the point such that

$|PA|^2 + |PB|^2 = |AB|^2$

$\Rightarrow (h-2)^2 + k^2 + (h+2)^2 + k^2 = 4^2 + 0$

$\Rightarrow 2h^2 + 8 + 2k^2 = 16$

$\Rightarrow h^2 + k^2 = 4$

$\therefore$ Locus of P is $x^2 + y^2 = 4$

15. Let the third vertex be (p, q).

$\Rightarrow q = p + 3$ ...(i)

Now, $\Delta = |5|$

$\Delta = \pm 5$

$$\frac{1}{2}\begin{vmatrix} p & q & 1 \\ 2 & 1 & 1 \\ 3 & -2 & 1 \end{vmatrix} = \pm 5$$

$\Rightarrow q + 3p - 7 = \pm 10$

$\Rightarrow 3p + q = 17$ ...(ii)

and $3p + q = -3$ ...(iii)

Solving Eqs. (i) and (ii) and solving Eqs. (i) and (iii), we get points $\left(\frac{7}{2}, \frac{13}{2}\right)$ and $\left(-\frac{3}{2}, \frac{3}{2}\right)$